Computer Engineering

Advanced Algorithms

1. Note that in parts (a) and (b) the values can be arranged in the heaps in several different ways, while still satisfying all the conditions required of a Binomial Heap. Your heaps will of course store the smallest values at the top, but you should explain how much flexibility there was beyond that and what policy you adopted in placing values.
    a. Sketch a Binomial Heap data structure containing the four values {2, 4, 6, 8}.
    b. Sketch a Binomial Heap storing the thirteen values {1, 3,..., 25}.

    [10 marks]
2. Form the union of the above two heaps, explaining the steps used and showing where the stored values end up. You do not need to display all the pointers in your data structures, and need not include any elaborate discussions of other operations on or applications of binomial trees or heaps. [10 marks]
3. Explain the steps involved in using the Miller-Rabin test to check whether a number N is composite. This will involve computing aN-1 mod N for some value of a. [10 marks]
4. Carry out the steps for N = 65 and a = 1, 2, 8 and 12. Comment on what (if anything) each partial result tells you about N and which cases (if any) help you to decide whether N is prime or what its factors might be.

    Pretend throughout the calculation that you do not know that 65 = 5 x 13. Proceed as though 65 were a huge number, imagining that you do not know at the outset whether it is prime or composite and that you are certainly unable to spot any factors. [10 marks]
5. A posting on a newsgroup announces the invention of a new compression algorithm, and claims that the method will guarantee to compress at least 10% of all possible input files by at least 10% of their original size, but that it might (unfortunately) cause some of the other 90% of possible inputs to expand by a factor of up to 90.

    Discuss how believable and reasonable the claim is. [8 marks]
6. Various compression utilities running on personal computers typically reduce text files to 1/3 of their original size. Estimate the proportion of all possible files (including binary ones) whose original length is less than 64 kbytes that could be compressed to this extent by an ideal compression algorithm. What proportion of the same set of files consists of

just alphanumeric characters, blanks and newlines? [12 marks]

7. Describe the structure of a Fibonacci heap and explain how to perform the following operations:
   a. create an empty heap
   b. add a single new node to a heap
   c. form the union of two existing heaps
   d. identify the smallest item in a heap

   How long does each of these operations take? [20 marks]
8. Suppose you are presented with a (large) integer N and are asked to find its complete factorization. You are not told anything at all in advance about how many factors it will have, but you are instructed to use the Pollard Rho method as a probabilistic algorithm as the core of your code. Explain
   i. The overall structure of the code you would write, where it calls Pollard Rho and any other sub-algorithms you will use, and what their purpose is. [4 marks]
   ii. What steps are taken in the Rho method and (informally) why it might be hoped that it will do what it is expected to. [6 marks]
   iii. The extents and manners in which parts of your code rely on random numbers and the consequences of these turning out to be either especially fortunate or especially awkward. [5 marks]
   iv. A coarse estimate of the total run-time for the factoriser in circumstances when the input number has exactly two large prime factors, and an equally crude estimate of the size that N would need to be before the factorization process took a whole day of CPU time on a modern desktop workstation. You may suppose that around 1013 basic operations are available in that amount of time. [5 marks]
9. Describe the structure of an ordinary heap, and document the costs associated with the following operations.
   a. Create a heap from n items where the items are all available at once but are initially in a random order.
   b. Remove the top (i.e., smallest) value stored in the heap.
   c. Given a pointer to an arbitrary item in the heap, re-instate the heap property after

the key associated with that single item is decreased in value.

d. Form a new heap whose elements are all those that are present in two other heaps (which may be destroyed in the combining process if that helps).

You are not expected to give detailed accounts of the algorithms involved. [6 marks]

10. Now explain the structure of a Binomial Heap and compare, with some explanation of your claims, the costs incurred in the same set of operations if Binomial rather than ordinary heaps were to be used. [14 marks]

11. Explain what is meant by the Kolmogorov Complexity K(n) of a natural number n. [5 marks]

Consider a graph of the function K(n) plotted against n:

i. Show that it is smooth, in the sense that for any n and fairly small value of k the value of K (n + k) will be quite close to the value of K(n). [3 marks]

ii. Show that it is rough, in the sense that for any N there are two values of ni and n2 between N and 2N such that K(m) is about 2K(n2), i.e. one has a complexity exponentially bigger than the other. [3 marks]

iii. Explain why the graph is bounded above by some straight line of the form n + c and comment on what the constant represents. [3 marks]

iv. Explain why for any constant k there will be a value N such that n > N implies that K(n) > k. [3 marks]

v. Demonstrate that there is no constant N such that n > N implies K(n) > log log log log n. [3 marks]

12. Explain how to check a large number for primality using a probabilistic method that gives you a bound of the probability of getting an incorrect judgment. [7 marks]

13. Give an asymptotic formula predicting the number of computer operations needed to verify that a number with n bits is prime, supposing that multiplication, division and remaindering are done using O(n2) methods and that you want to achieve a probability of error bounded by 1 in 260. You do not need to prove that the algorithm you describe works, but you should nevertheless explain it carefully and completely. [7 marks]

14. The gap between adjacent primes near the integer N is roughly log(N). Estimate roughly the number of computer operations you would expect to be needed to find a 2000-bit prime that is just slightly larger than some given 2000-bit random number. [6 marks]

15. State the fundamental theorem of linear programming. [3 marks]

16. Consider the following linear program:

Minimize $-3x_1 - 2x_2$

subject to:

$$3x_1 + x_2 \leq 5$$
$$-2x_1 \geq -10 + 4x_2$$
$$x_1 . x_2 \geq 0.$$

(i) Convert this LP into standard and slack form, and specify the initial basic solution. [4 marks]

(ii) Solve this LP using the simplex algorithm. Specify the associated basic solution after each iteration. [4 marks]

17. We consider the Steiner Tree Problem defined as follows. We are given an undirected, connected graph G = (V, E) with a non-negative cost-function c : E ^ R+. Further, we are given a set S C V of terminals. The goal is to find a minimum-cost subgraph of G that connects all terminals, where the cost of a subgraph is the sum of the costs of its edges. Consider the following algorithm:

- Let H = (V, E') be the metric completion of G, where E' = {{u,v}: u,v E V} and c({u,v}) is the cost of the shortest path from u to v in G.
- Compute a Minimum Spanning Tree T on the subgraph H [S] induced by the set of terminals S.
- Replace every edge {u, v} in T by the edges of a shortest path from u to v in G, and return the solution.

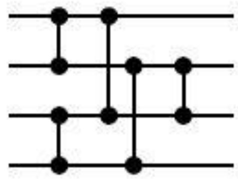
(i) Prove an upper bound of $2\left(1 - \frac{1}{|S|}\right)$ on the approximation ratio of this algorithm. [Hint: You can use an approach similar to the analysis of Approx-TSP- Tour.] [6 marks]

(ii) Construct an example which provides a matching lower bound on the approximation ratio. [3 marks]

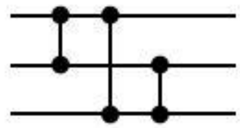
18. State the zero-one principle in the context of sorting networks. [2 marks]

19. For each of the following six comparison networks, state whether it is a sorting network or not. In each case, justify your answer. For the justification you may refer to standard

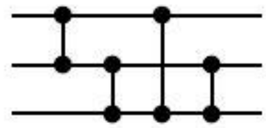
results without giving a proof. [9 marks]



(1)



(2)



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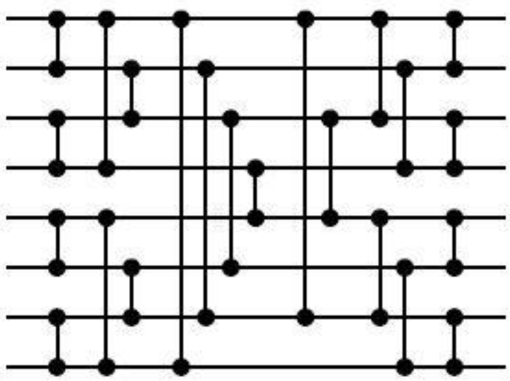
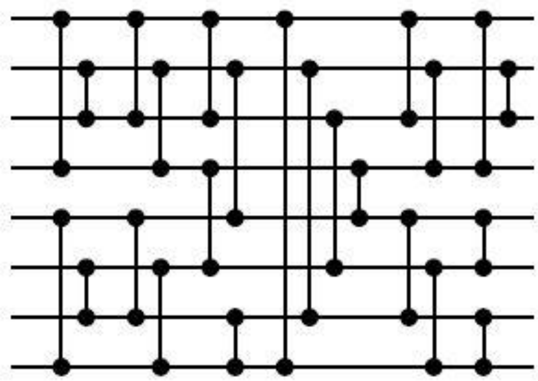
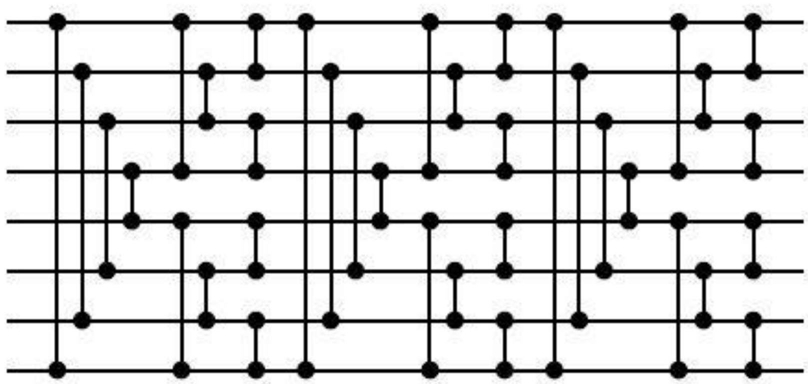


(4)



(5)



(6)

20. Let n be an exact power of 2. Show how to construct an n-input, n-output comparison network of depth log n in which the top output wire always carries the minimum input value and the bottom output wire always carries the maximum input value. [4 marks]
21. (i) Prove that the number of comparators in any sorting network is Q (n log n). [4 marks]
    (ii) What does Part (d)(i) imply in terms of the depth of any sorting network? [1 mark]